ON THE HAMILTONIAN FORMULATION OF CLASS B BIANCHI COSMOLOGICAL MODELS *

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Abstract

The application of the divergence theorem in a non-coordinated basis is shown to lead to a corrected variational principle for Class B Bianchi cosmological models. This variational principle is used to construct a Hamiltonian formulation for diagonal and symmetric vacuum Class B models. [Note: This is an unpublished paper from 1984 that might be useful to anyone interested in Class B Bianchi models].

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I. INTRODUCTION

For the past decade there has been a low-level but continuing interest in Bianchi-type cosmological models which admit a cosmic time, a gauge in which $g_{0i} = 0$, and whose t = const. surfaces are homogeneous three-spaces of Bianchi types I-IX [1]. This interest, for the most part, centers on the mathematical structure of the Einstein equations for these models instead of on astrophysical applications, since these models could only represent an unobservable portion of the actual universe (although a search for observable consequences of this primitive stage continues). The advantage of the Bianchi-type models is that they provide a model of general relativity which has true dynamics, even though the metric components in the invariant basis are fuctions of time only. These models allow one to study problems in general relativity in a situation where they are soluble exactly (or at least qualitatively).

One line of study has been the Hamiltonian formulation of the Einstein equations for these models which was begun with an eye toward approximate solutions and the construction of a model quantum theory for general relativity. The Hamiltonian formulation for Types I and IX based on the Arnowitt-Deser Misner [2] (ADM) formulation of general relativity is due to Misner [3], and was extended to all models of Ellis-MacCallum [4] Class A by Ryan [5]. In Ref. [5] an extension was made to Class B models, but MacCallum and Taub [6] pointed out that the naive application of the ADM formalism yields a Hamiltonian formulation that gives incorrect Einstein equations for these models. This pathology excited some interest, and a number of corrected variational principles have been proposed for these models [7,8,9,10]

In this paper we plan to make a detailed review of the problem of the Hamiltonian

formulation of Class B models from a slightly different point of view. All of the variational principles in Refs. [7-10] have what Jantzen [10] calls "non-potential" terms in their variational principles, and there is at least one other problem that these models have that makes the study of this Hamiltonian formulation interesting. We plan to emphasize that the non-potential terms depend on the treatment of certain terms in the Einstein action, and that no extra correction terms are needed. In fact, with the proper treatment of the divergence theorem in a non-coordinated basis, the correction becomes obvious. This correction provides a foundation for the idea of one of us [11] that the problem of Class B models is due to problems of variational principles in a non-coordinated basis, although not in the form originally proposed.

A model problem for Class B models that suffers from all the same problems (although in a slightly different form) is that of conformal metrics in general relativity. Consider the metric

$$ds^2 = e^{2\lambda(x,t)}(\eta_{\mu\nu} dx^\mu dx^\nu). \tag{1.1}$$

The Einstein tensor is

$$G_{\mu\nu} = 2\partial_\mu\partial_\nu\lambda - \partial_\mu\lambda\partial_\nu\lambda - 2\eta_{\mu\nu}(\eta^{\alpha\beta}\partial_\alpha\partial_\beta\lambda + \frac{1}{4}\eta^{\alpha\beta}\partial_\alpha\lambda\partial_\beta\lambda). \tag{1.2}$$

The Einstein action in vacuum becomes

$$I = \frac{1}{16\pi}\int \sqrt{-g}R\, d^4x = \frac{1}{8\pi}\int e^{2\lambda}[6\eta^{\alpha\beta}\partial_\alpha\partial_\beta\lambda + 4\eta^{\alpha\beta}\partial_\alpha\lambda\partial_\beta\lambda]d^4x. \tag{1.3}$$

Here it is easy to see one of the pathologies we will see in Class B models: *Reduced actions may lack equations.* Varying (1.3) with respect to λ gives only one equation, and setting $G_{\mu\nu}$ from (1.2) equal to zero gives nine more, and only in cases where these nine are automatically zero or redundant (as in $k = 0$ FRW models) will (1.3) yield the correct Einstein equations.

The second difficulty that we will encounter in Class B models can be illustrated by rewriting the action (1.3) in an orthonormal basis, $\mathbf{e}_\mu = e^{-\lambda}(\partial/\partial x^\mu)$, $\omega^\mu = e^\lambda dx^\mu$. We will write the action of $\mathbf{e}_\mu$ on a function A as $\mathbf{e}_\mu A \equiv A_{,\mu} = e^{-\lambda}\partial_\mu A$. The volume element $d^4x = e^{-4\lambda}\omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3$, so the action becomes

$$I = \frac{1}{8\pi}\int [6\eta^{\alpha\beta}\lambda_{,\alpha,\beta} + 10\eta^{\alpha\beta}\lambda_{,\alpha}\lambda_{,\beta}]\omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3. \tag{1.4}$$

If one treats the derivatives $\mathbf{e}_\mu$ like ordinary derivatives and varies (1.4), one finds as the vacuum equations

$$\eta^{\alpha\beta}\lambda_{,\alpha,\beta} = 0 = \eta^{\alpha\beta}\partial_\alpha\partial_\beta\lambda - \eta^{\alpha\beta}\partial_\alpha\lambda\partial_\beta\lambda. \tag{1.5}$$

Varying (1.3) one finds

$$6\eta^{\alpha\beta}\partial_\alpha\partial_\beta\lambda + 4\eta^{\alpha\beta}\partial_\alpha\lambda\partial_\beta\lambda.$$

The obvious problem is treating $\mathbf{e}_\mu$ derivatives as though they were partial derivatives in integration by parts. This is the second problem encountered in Class B models: *Integration by parts in non-coordinated bases can lead to incorrect equations.* In Sec. II we will show that the answer to this problem lies in the proper use of the divergence theorem in a non-coordinated basis based on the model of Spivak [12] or Lovelock and Rund [13].

The main thrust of this paper is to show how the application of the correct divergence thoerem can help in the understanding of the non-potential terms in the Hamiltonian formulation of Class B models. In fact, the only non-potential terms are those that arise from the derivatives of the connection coefficients in the Ricci scalar in the Einstein variational principle. As a practical excercise we apply the results to diagonal and symmetric [14] Class B metrics. While we feel that our presentation is more didactic, some of the results for diagonal and symmetric models can be found in the exhaustive review of Jantzen [10].

Note that Jantzen manages to achieve a Hamiltonian formulation without non-potential terms valid for certain metric variables, at the cost (admmittedly slight) of introducing g_{0i} terms which must be found by integrating supplementary equations.

The plan of the paper is as follows. In Sec. II we discuss the divergence theorem in a non-coordinated basis and its application to the Hamiltonian formulation of Class B models. In Sec. III we consider vacuum Class B models with diagonal space metrics, showing which of them allow solutions. Certain models do not, and in Sec. IV we show that all models with symmetric metrics do allow vacuum solutions.

II. THE DIVERGENCE THEOREM IN A NON-COORDINATED BASIS AND THE VARIATIONAL PRINCIPLE FOR CLASS B MODELS.

As mentioned in the introduction the corrected variational principles that have been proposed for Class B models [7-10] have non-potential terms (with the exception of the final Hamiltonian of Jantzen [10]). The first calculation of correction terms was due to Sneddon [7], who called, the non-potential terms "surface terms". Our approach is slightly different, and, we feel, somewhat more didactic.

The basic cause of the problem for Class B models is that the divergence theorem in a non-coordinated basis [12,13] does not have the usual form. If we take, for example, the formulation of Lovelock and Rund [13], we find that the divergence theorem, a special case of Stoke's theorem, takes the following form in an n-dimensional space. If we have a vector $A = A^i \mathbf{e}_i$, the divergence theorem is given in terms of an $(n-1)-$ form π_j defined by

$$\pi_j \equiv (-1)^{j+1} \omega^1 \wedge \cdots \wedge \omega^{j-1} \wedge \omega^{j+1} \wedge \cdots \wedge \omega^n \tag{2.1}$$

Stokes theorem for the form $\sigma = \pi_j A^j$ is

$$\int_G d\sigma = \int_{\partial G} \sigma. \tag{2.2}$$

If we calculate $d\sigma$ we find

$$d\sigma = A^i_{,i}\omega^1 \wedge \cdots \wedge \omega^n + A^j(-1)^{j+1} d\omega^1 \wedge \cdots \wedge \omega^{j-1} \wedge \omega^{j+1} \wedge \cdots \wedge \omega^n$$

$$+ \cdots + A^j(-1)^{j+1}\omega^1 \wedge \cdots \wedge \omega^{j+1} \wedge \cdots \wedge d\omega^n,$$

where $A^i_{,j} \equiv \mathbf{e}_j A^i$. Using the fact that in the terms containing A^j and $d\omega^k$ only the terms $\frac{1}{2}C^k_{kj}\,\omega^k \wedge \omega^j$ give a non-zero contribution, we find that

$$d\sigma = (A^i_{,i} + A^i C^l_{jl})\omega^1 \wedge \cdots \wedge \omega^n. \tag{2.3}$$

This means that

$$\int_G A^i_{,i}\omega^1 \wedge \cdots \wedge \omega^n = \int_G -A^i C^l_{jl}\omega^1 \wedge \cdots \wedge \omega^n + \int_{\partial G} \sigma \tag{2.4}$$

It is a moot point whether first term on the right-hand-side of (2.4) should be called a "surface term" or not. Here we have chosen to say that only the integral over ∂G is to be regarded as a surface term.

We would now like to apply the above form of the divergence theorem to Class B Bianchi models. If we write the Einstein action for Bianchi models in the basis $(d\tau, \omega^1, \omega^2, \omega^3)$, where τ is a time we choose and $\{\omega^i\}$ is the invariant basis, we find that

$$I = \frac{1}{16\pi}\int {}^4R\sqrt{-{}^4g}\, d\tau \wedge \omega^1 \wedge \omega^2 \wedge \omega^3, \tag{2.5}$$

where ${}^4g$ is the determinant of the metric in the above basis. The usual ADM reduction gives

$$I = \frac{1}{16\pi}\int [\pi^{ij}\dot{g}_{ij} - N\mathcal{H}_\perp - N_i\mathcal{H}^i] d\tau \wedge \omega^1 \wedge \omega^2 \wedge \omega^3, \tag{2.6}$$

where the π^{ij} are treated as an independent momenta, g_{ij} is the metric on τ = const. surfaces in the basis $\{\omega^i\}, N_i = g_{0i}$, and $N = 1/(-g^{00})^{1/2}$. The Hamiltonian and space constraints are

$$\mathcal{H}_\perp \equiv -\sqrt{g}\{R + g^{-1}[\frac{1}{2}(\pi^k k)^2 - \pi^{ij}\pi_{ij}]\},$$

$$\mathcal{H}^i \equiv -2(\pi^{ij}_{,j} + \Gamma^i_{jk}\pi^{jk} + \Gamma^{ij}_{kj}\pi^{ik} - \Gamma^k_{kj}\pi^{ij}), \tag{2.8}$$

where, as usual, three-dimensional indices are raised and lowered with g_{ij} and $g^{ij} \equiv [g_{ij}]^{-1}$, $g \equiv \det[g_{ij}]$, comma i means operation with $\mathbf{e}_i$, the invariant vector dual to ω^i, and R is the three-dimensional Ricci scalar [15],

$$g^{rs}(\Gamma^l_{rs})_{,l} - g^{rs}(\Gamma^l_{rl})_{,s} + g^{rs}\Gamma^t_{rs}\Gamma^l_{tl} - g^{rs}\Gamma^t_{rl}\Gamma^l_{ts} - g^{rs}C^t_{ls}\Gamma^l_{rt}. \tag{2.9}$$

We would like to assume that the Bianchi models are spatially homogeneous, that is that all quantities appearing in the action have no space derivatives. Because of the changed divergence theorem this is not possible, so we will attempt a "maximal homogenization" compatible with Class B models. Spatial derivatives appear in three places: i)$\pi^{ij}_{,j}$, ii) $\Gamma^i_{jk,l}$, and iii) the spatial derivatives of the g_{ij} that appear in the expresion for $\Gamma^i_{jk,}$

$$\Gamma^i_{jk} = \frac{1}{2}g^{il}(g_{lj,k} + g_{lk,j} - g_{jk,l})$$

$$+\frac{1}{2}(C^i_{jk} - g_{jp}g^{in}C^p_{nk} - g_{pk}g^{in}C^p_{nj}). \tag{2.10}$$

We will show that is possible to ignore the $g_{ij,k}$ in the $\Gamma^i_{jk,}$ and use in the final action only $\tilde{\Gamma}^i_{jk} = \frac{1}{2}(C^i_{jk} - g_{jp}g^{in}C^p_{nk} - g_{pk}g^{in}C^p_{nj})$. From the form of the action (2.7) and of $\mathcal{H}^i$, it is obvious that the term with $\pi^{ij}_{,j}$ will cause no problem in any gauge in which $N_i = 0$, and we will always work in such a gauge. All of this means that only the terms in R that depend on derivatives of the Γ^i_{jk} will cause problems. It has been shown [11] that the problem with R without derivative terms is that $\delta(\sqrt{g}R)/\delta g_{ij} \neq -\sqrt{g}R^{ij} + \frac{1}{2}\sqrt{g}g^{ij}R$. We

will show that if R is given by (2.9) and we use the divergence theorem (2.5), that indeed $\delta(\sqrt{g}R)/\delta g_{ij} = -\sqrt{g}R^{ij} + \frac{1}{2}\sqrt{g}g^{ij}R$. We will use $\tilde{R}$ to denote R written with $\tilde{\Gamma}^i_{jk}$ in place of Γ^i_{jk}. If we set the derivative terms in R equal to zero we find that

$$R(\Gamma, = 0) = -\frac{1}{2}C^t_{si}C^s_{tj}g^{ij} - \frac{1}{2}C^a_{si}C^b_{tj}g^{st}g_{ab}g^{ij}$$
$$+\frac{1}{4}g^{st}g^{pr}C^j_{sp}C^b_{tr}g_{bj} + C^s_{sr}C^k_{tk}g^{rt}. \tag{2.11}$$

When we vary R we will homogenize after variation, so after variation all derivative terms will be put equal to zero. If in (2.9) we replace Γ^i_{jk} by $\tilde{\Gamma}^i_{jk}$, we can see that the term $g^{rs}(\tilde{\Gamma}^l_{rl})_{,s}$ will not contribute to the variation since $\tilde{\Gamma}^l_{rl} = C^l_{rl}$ and is independent of g_{ij}. Variation now gives

$$\frac{\delta(\sqrt{g}R)}{\delta g_{ij}} = \frac{1}{2}\sqrt{g}g^{ij}R + \sqrt{g}\left(\frac{\delta R[\Gamma, = 0]}{\delta g_{ij}}\right) - \sqrt{g}g^{rs}C^n_{ln}\left(\frac{\delta\tilde{\Gamma}^l_{rs}}{\delta g_{ij}}\right). \tag{2.12}$$

Using (2.11) and the definition of $\tilde{\Gamma}^i_{jk,}$ it is not difficult to show that

$$\frac{\delta(\sqrt{g}R)}{\delta g_{ij}} = \frac{1}{2}\sqrt{g}g^{ij}R - \sqrt{g}R^{ij}, \tag{2.13}$$

and since in the spatially homogeneous case $\tilde{R} = R$ we see that our assertion that variation of R using the proper divergence theorem gives the correct Einstein equations.

In the following two sections we will apply the above formalism to the simplest classes of vacuum models and construct a Hamiltonian formalism. In our case we can follow the outline given in Ryan and Shepley [15], where we use the parametrization

$$g_{ij} = e^{2\Omega}(e^{2\beta})_{ij}, \tag{2.14}$$

where $\Omega = \Omega(t)$ and the matrix β_{ij} is

$$\beta = e^{-\psi\kappa^3}e^{-\theta\kappa^1}e^{-\phi\kappa^3}\beta_d e^{\phi\kappa^3}e^{\theta\kappa^1}e^{\psi\kappa^3} \tag{2.15}$$

where

$$\kappa^3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \kappa^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \tag{2.16}$$

and

$$\beta_d = \mathrm{diag}(\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+). \tag{2.17}$$

we also define a matrix p_{ij} as

$$p_{ij} \equiv 2\pi(e^{\beta}_{is}\pi^s_t e^{-\beta}_{tj}) - \frac{2\pi}{3}\delta_{ij}\pi^k_k, \tag{2.18}$$

where p_{ij} is a function of $(p_\pm, p_\phi, p_\psi, p_\theta, \beta_\pm, \theta, \phi, \psi)$ such that

$$I = \int [p_+ d\beta_+ + p_- d\beta_- + p_\psi d\psi + p_\phi d\phi + p_\theta d\theta - H d\Omega]. \tag{2.19}$$

The Hamitonian H is a function of $(p_\pm, p_\phi, p_\psi, p_\theta, \beta_\pm, \theta, \phi, \psi)$ equal to $2\pi(\pi^k_k)$ and is obtained by solving $\mathcal{H}_\perp = 0$. In those cases where the $\mathcal{H}^i$ are not automatically zero, we add the additional constraints $\mathcal{H}^i = 0$. It is not difficult to show that for all Class B models $\mathcal{H}^i \neq 0$ and

$$H^2 = 6p_{ij}p_{ij} - 24\pi^2 e^{-6\Omega}\tilde{R}. \tag{2.20}$$

The "potential" term now has two parts, $R(\Gamma, = 0)$ and $g^{rs}(\tilde{\Gamma}^l_{rs,l})$ and the Einstein equations are Hamilton's equations in the following form:

$$\dot{q} = \delta H/\delta p, \qquad \dot{p} = -\delta H/\delta q, \tag{2.21}$$

where $\cdot \equiv d/d\Omega$ and the δ derivatives include integration by parts by means of the correct divergence theorem. An important point is that the term $g^{rs}(\Gamma^l_{rs,l})$ can never be converted into a pure potential term, because the part of the variation of the form $\delta g^{rs}(\Gamma^l_{rs,l})$ is set equal to zero, while $g^{rs}(\delta\Gamma^l_{rs,l})$ gives a non zero term. There does not exist any pure

function of g_{ij} which has this property. Note also that a good test of the correctness of (2.21) will be $(\mathcal{H}^i)^{\cdot} = 0$ as an identity.

In the following two sections we shall study the diagonal (β diagonal) and symmetric cases (β with one off-diagonal term) to show effect of the term $g^{rs}(\Gamma^l_{rs,l})$ on Hamilton's equations.

III. DIAGONAL VACUUM CLASS B MODELS

It is worthwhile to consider the vacuum diagonal Class B models as a cautionary tale, because they are a zoo of pathology. In principle it is easy to write the corrected Hamiltonian for the Class B models. We have $p_{ij} = p_+\alpha_1 + p_-\alpha_2$, where

$$\alpha_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \qquad \alpha_2 = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & -\sqrt{3} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.1}$$

The squared Hamiltonian becomes

$$H^2 = p_+^2 + p_-^2 + 12\pi^2 e^{-4\Omega} V(\beta_\pm) - 24\pi^2 e^{-6\Omega} g^{rs}(\tilde{\Gamma}^l_{rs,l}). \tag{3.2}$$

We use the form of the Ellis-MacCallum [4] scheme for writing the constant C^i_{jk} given in Ryan and Shepley [15]. That is,

$$C^i_{jk} = \varepsilon_{jks} m^{si} + \delta^i_k a_j - \delta^i_j a_k.$$

For all Class B Models $a_i = (1/2)C^l_{il} \propto \delta^3_i$, so only the term $g^{rs}(\tilde{\Gamma}^3_{rs})_{,3}$ contributes to H^2. Since $m^{ij}a_j = 0$ by the Jacobi identity, $m^{i3} = 0$ for all i. If we define the *spin coefficients* of the matrix m^{AB}, $A, B = 1, 2$ as $\alpha = \frac{1}{2}(m^{11} + m^{22})$, $\lambda = -\frac{1}{2}(m^{11} - m^{22})$, $\gamma = m^{12}$, the Hamiltonian (3.2) reduces to

$$H^2 = p_+^2 + p_-^2 + 12\pi^2 e^{-4\Omega} e^{4\beta_+}[2(\alpha^2 + \lambda^2)\cosh(4\sqrt{3}\beta_-)$$

$$+4\lambda\alpha \sinh\,(4\sqrt{3}\beta_-) - 2\alpha^2 + 2\lambda^2 + 4\gamma^2 + 12a_3^2] - 24\pi^2 e^{-4\Omega} e^{4\beta_+}[-12a_3(\beta_+)_{,3} - 4\sqrt{3}\gamma(\beta_-)_{,3}], \tag{3.4}$$

where we have replaced $\tilde{\Gamma}^l_{rs}$ by its expression in terms of g_{ij} in the parametrization (2.14) and taken the necessary derivatives. Notice that the first of the derivative terms depends only on β_+, and this allows one to integrate by parts. That is, using the correct divergence theorem, variation of the part of H containing a_3 gives

$$\frac{\delta H}{\delta \beta_+}(\alpha = \lambda = \gamma = 0) = \frac{1}{H}\{576\pi^2 e^{-4\Omega} e^{4\beta_+} a_3^2 - 576\pi^2 e^{-4\Omega} e^{4\beta_+} a_3^2\} = 0, \tag{3.5}$$

so a reduced Hamiltonian for all diagonal Class B models which gives the equations of motion for $\beta_\pm$ (i.e. $\dot{H} \neq \partial H/\partial\Omega$) is given by

$$H^2 = p_+^2 + p_-^2 + 12\pi^2 e^{-4\Omega} e^{4\beta_+}[2(\alpha^2 + \lambda^2)\cosh(4\sqrt{3}\beta_-) + 4\lambda\alpha \sinh(4\sqrt{3}\beta_-) -$$

$$-2\alpha^2 + 2\lambda^2 + 4\gamma^2] + 96\sqrt{3}\pi^2 e^{-4\Omega} e^{4\beta_+}\gamma(\beta_-)_{,3}. \tag{3.6}$$

This reduction was also noticed by Jantzen [10]. No further integration by parts is possible, because variation of the remaining derivative term with respect to β_+ must give zero and with respect to β_- must give $-192\sqrt{3}\pi^2 e^{-4\Omega} e^{4\beta_+}\gamma$.

To study this Hamiltonian we give α, λ, γ, and a_3 in Table I. It would *seem* at this point that Types V and VI give simple Hamiltonians because $\gamma = 0$. However, it is instructive to apply the test suggested in the previous section, that is, $(\mathcal{H}^i)^{\cdot} = 0$. For all the Class B diagonal models $\mathcal{H}^1$ and $\mathcal{H}^2$ are identically zero and $\mathcal{H}^3 = 0$ reduces to

$$6a_3 p_+ + 2\sqrt{3}\gamma p_- = 0 \tag{3.7}$$

Table I

Bianchi Type	α	λ	γ	a_3
III	0	0	$\frac{1}{2}$	$-\frac{1}{2}$
IV	$\frac{1}{2}$	$-\frac{1}{2}$	0	-1
V	0	0	0	-1
$\mathrm{VI}_{h\neq-1}$	0	0	$\frac{1}{2}(h-1)$	$-\frac{1}{2}(h+1)$
$\mathrm{VII}_{h\neq0}$	-1	0	$\frac{h}{2}$	$-\frac{h}{2}$

It is easy to see that Type V models with $\alpha = \gamma = 0$ are consistent. The reduced Hamiltonian is

$$H^2 = p_+^2 + p_-^2, \tag{3.8}$$

and $p_+ = 0$ is the space constraint. It is even possible to construct a Hamiltonian that obeys $\dot{H} = \partial H/\partial\Omega$, that is, $p_+ = 0$ implies $\beta_+ = \beta_+^0$ and a final Hamiltonian is

$$H^2 = p_-^2 + 12\pi^2 e^{-4\Omega} e^{4\beta_+^0}. \tag{3.9}$$

This Hamiltonian gives as a general solution for vacuum diagonal Type V models

$$\beta_+ = \beta_+^0$$

$$p_- = \text{const.}$$

$$\beta_- = \beta_-^0 + p_-\Omega$$

$$H = \{p_-^2 + 12\pi^2 e^{-4\Omega} e^{4\beta_+^0}\}^{1/2} = \frac{1}{12\pi} e^{3\Omega} \left(\frac{d\Omega}{dt}\right)^{-1} \tag{3.10}$$

Let us now examine the Type IV case. If we calculate $6a_3\dot{p}_+$ $(\gamma = 0)$ we find $6a_3\dot{p}_+ = 6a_3\left[48\pi^2 e^{-4\Omega} e^{4\beta_+}\{\cosh(4\sqrt{3}\beta_-) \;-\sinh(4\sqrt{3}\beta_-)\}\right] \neq 0$!

This equation is one of the reasons we have qualified the diagonal vacuum Class B models as a zoo of pathology. A check of $(\mathcal{H}^i)^{\cdot} = 0$ shows that models of types III, V, and $\mathrm{VI}_{h\neq-1}$ are consistent, that is those for which $\alpha = \lambda = 0$. The two remaining types, IV and $\mathrm{VII}_{h\neq 0}$, are examples of another problem in reduced variational principles. We have written our variational principle for diagonal metrics by inserting a diagonal metric directly into the Einstein action. The problem here is that if the R_{ij}, $(i \neq j)$ are not identically zero, we will lose the equations $R_{ij} = 0$ $(i \neq j)$. A rapid calculation shows that R_{ij} is diagonal for a diagonal metric for types III, V, and $\mathrm{VI}_{h\neq-1}$, while for types IV and $\mathrm{VII}_{h\neq 0}$, $R_{12} \neq 0$. This seems to indicate that symmetric metrics $(g_{12} \neq 0)$ will always provide consistent vacuum equations if such metrics always have $R_{13} = R_{23} = 0$ identically. In the following section we show that this is indeed the case, and present a consistent Hamiltonian formulation for all symmetric vacuum Class B models.

IV. THE SYMMETRIC CASE

In the previous section we have shown that the corrected Hamiltonian as given by (2.20) does not always yields equations of motion which are consistent with the space constraint equation when one assumes that the metric of the homogeneous hypersurface is diagonal. This is certainly the case with diagonal metrics that admit groups of isometries of Bianchi types IV and $\mathrm{VII}_{h\neq 0}$. We found out that a common feature appeared in those cases where the Hamiltonian equations of motion were not consistent with the space constraint. Namely, that some of the non-diagonal components of the Ricci tensor do not vanish as one might expect from the diagonality of the space metric.

In this Section we shall consider symmetric metrics (one nonzero off-diagonal element

in the metric) and show that all the components of the Ricci tensor for which the corresponding metric component is zero vanish in this case. Therefore the "pathology" found with diagonal metrics is not present in the symmetric case, which tells us that the corrected Hamiltonian for symmetric metrics must provide the precise equations of motion that are consistent with the space constraint equation. This is formally shown by writing explicity the corrected Hamiltonian for symmetric metrics and the corresponding Hamilton equations. Then these equations are used along with the space constraint equation to show that the latter is fulfiled at all times, and therefore that the equations of motion are consistent with the space constraint equation.

As we are assuming a space symmetric metric, we set $g_{13} = g^{13} = g_{23} = g^{23} = 0$ and then we have only to compute the R_{13} and R_{23} components of the Ricci tensor in the order to find out whether they vanish or not. The Ricci tensor is given by

$$R_{ij} = \frac{1}{2}a_i a_j + 2g_{ij}g^{kt}a_k a_t + g_{ik}g^{sb}\varepsilon_{sjl}m^{lk}a_b$$
$$+g_{jk}g^{sb}a_b\varepsilon_{sil}m^{lk}a_b - 1/2\varepsilon_{sjt}\varepsilon_{kin}m^{tk}m^{ns}$$
$$-\frac{1}{2}\varepsilon_{sjt}\varepsilon_{kin}g^{sk}g_{ab}m^{tb}m^{na}, \tag{4.1}$$

with a_i, m^{ij}, g_{ij} and ε_{ijk} as defined previously. Then immediately follows that the R_{13} component is

$$R_{13} = g_{3k}g^{33}m^{2k}a_3^2 - \frac{1}{2}\varepsilon_{13t}\varepsilon_{k1n}m^{tk}m^{n1} - \frac{1}{2}\varepsilon_{23t}\varepsilon_{k1n}m^{tk}m^{n2} -$$
$$-\frac{1}{2}\varepsilon_{13t}\varepsilon_{k1n}m^{tb}m^{na} - \frac{1}{2}\varepsilon_{23t}\varepsilon_{k1n}g^{2k}g_{ab}m^{tb}m^{na}, \tag{4.2}$$

where we have used the fact that $a_1 = a_2 = 0$ (see Ref. [15]), $g_{13} = 0$ and the symmetry and skewsymmetry of g^{ij} and ε_{ijk} respectively. Expression (4.2) can be reduced to

$$R_{13} = 1/2\varepsilon_{k1n}g^{1k}g_{ab}m^{2b}m^{na},$$

and it is straightforward to show that this $R_{13}=0$.

Now we compute the R_{23} component of the Ricci Tensor. By the same arguments (viz. $a_1=a_2=0$, $g_{23}=0$, g_{ij} and $\varepsilon_{ijk}=-\varepsilon_{jik}$) from (4.1) we obtain

$$R_{23}=g_{3k}g^{s3}\varepsilon_{s2l}m^{lk}a_3^2-\varepsilon_{13t}\varepsilon_{k2n}m^{tk}m^{n1}-\varepsilon_{23t}\varepsilon_{k2n}m^{tk}m^{n2}$$

$$-\frac{1}{2}\varepsilon_{13t}\varepsilon_{k2n}g^{lk}g_{ab}m^{tb}m^{na}-\frac{1}{2}\varepsilon_{23t}\varepsilon_{k2n}g^{2k}g_{ab}m^{tb}m^{na}. \tag{4.4}$$

Imposing $m^{13}=m^{31}=m^{23}=m^{32}=0$ and $g_{13}=g_{23}=0$, (4.4) becomes

$$R_{23}=\varepsilon_{k2n}m^{2k}m^{n1}-\varepsilon_{k2n}m^{1k}m^{n2}, \tag{4.5}$$

or simply $R_{23}=0$.

We should point out that the above results, namely $R_{13}=0$ and $R_{23}=0$, hold for all symmetric metrics which admit groups of isometries of Class B.

The form of g_{ij} for the symmetric case is (2.14) and (2.15) with $\psi=\theta=0$. The corrected Hamiltonian for this case is

$$H^2=p_+^2+p_-^2+\frac{3p_\phi^2}{\sinh(2\sqrt{3}\beta_-)}-24\pi^2e^{-6\Omega}\tilde{R}, \tag{4.6}$$

where $\tilde{R}$, as before, is given by

$$\tilde{R}=R+g^{rs}(\tilde{\Gamma}^l_{rs})_{,l}, \tag{4.7}$$

with the Ricci (curvature) scalar defined by (2.11) and the homogenized connections as defined in Sec. II.

As we did in the diagonal case, one can show that the only contribution from $g^{rs}(\tilde{\Gamma}^l_{rs})_{,l}$ comes from the term $g^{rs}(\tilde{\Gamma}^3_{rs})_{,3}$, that is,

$$g^{rs}(\tilde{\Gamma}^l_{rs})_{,l}=-a_3[g^{11}(g^{33}g^{11})_{,3}+g^{22}(g^{33}g_{22})_{,3}$$

$$+2g^{12}(g^{33}g_{12})_{,3}] - g^{11}[g^{33}g_{1p}m^{2p}]_{,3}$$

$$+g^{22}[g^{33}g_{2p}m^{1p}] + g^{12}[g^{33}g_{1p}m^{1p}]_{,3}$$

$$-g^{12}[g^{33}g_{2p}m^{2p}]_{,3}. \tag{4.8}$$

Of course we have used $g_{13} = g_{23} = g^{23} = 0, a_1 = a_2 = 0$ and the symmetry of m^{ij}.

Now we proceed to obtain the explicit form of (4.8) in terms ot the parametrization (2.14 - 2.15). However, in order to have a compact expression for the final result we introduce the spin coefficients α, λ, γ of the rotated matrix $e^{\phi\kappa^3}\tilde{m}e^{-\phi\kappa^3}$, where $\tilde{m}$ is given by

$$\tilde{m} = \begin{bmatrix} m^{11} & m^{12} \\ m^{21} & m^{22} \end{bmatrix}$$

We find

$$\alpha = \frac{1}{2}(m^{11} + m^{22}), \tag{4.9}$$

$$\lambda = \frac{1}{2}\cos 2\phi(m^{11} - m^{22}) + \sin 2\phi m^{12}, \tag{4.10}$$

and

$$\gamma = -\frac{1}{2}\sin 2\phi(m^{11} - m^{22}) + \cos 2\phi m^{12}. \tag{4.11}$$

After a lengthy algebraic computation we end up with the following result for the expression (4.8),

$$g^{rs}(\tilde{\Gamma}^l_{rs})_{,l} = e^{2\Omega}e^{4\beta+}[-12a_3\beta_{+,3} - 4\sqrt{3}\gamma\beta_{-,3} - 2\alpha\phi_{,3}$$

$$+2\lambda\sinh(4\sqrt{3}\beta_-)\phi_{,3} + 2\alpha\cosh(4\sqrt{3}\beta_-)\phi_{,3}]. \tag{4.12}$$

The scalar curvature R has the same form as in the diagonal case with α, γ, λ replaced by (4.9 - 4.11), so the final Hamiltonian becomes

$$H^2 = p_+^2 + p_-^2 + \frac{3p_\phi^2}{\sinh^2(2\sqrt{3}\beta_-)} + 24\pi^2 e^{-4\Omega}e^{4\beta+}[(\alpha^2 + \lambda^2)\cosh(4\sqrt{3}\beta_-) +$$

$$+2\alpha\lambda\sinh(4\sqrt{3}\beta_-)-(\alpha^2-\lambda^2)+2\gamma^2+6a_3^2]-$$

$$-24\pi^2e^{-4\Omega}e^{4\beta+}[-12a_3\beta_{+,3}-4\sqrt{3}\gamma\beta_{-,3}-2\alpha\phi_{,3}$$

$$+2\lambda\sinh(4\sqrt{3}\beta_-)\phi_{,3}+2\alpha\cosh(4\sqrt{3}\beta_-)\phi_{,3}]. \tag{4.13}$$

To compute the space constraint equations for this symmetric case we make use of the expression (2.8) and impose $g_{13}=g^{13}=g_{23}=g^{23}=0$ and $\pi_{13}=\pi^{13}=\pi_{23}=\pi^{23}=0$. Equation (2.8) can be broken down into

$$g^{1t}\pi^k_s C^s_{tk}-C^j_{kj}\pi^{1k}=0, \tag{4.14}$$

$$g^{2t}\pi^k_s C^s_{tk}-C^j_{kj}\pi^{2k}=0, \tag{4.15}$$

and

$$g^{3t}\pi^k_s C^s_{tk}-C^j_{kj}\pi^{3k}=0 \tag{4.16}$$

If one uses the expression in Sec. II for the structure coefficients, then by virtue of the symmetric form of g_{ij} and π^{ij} Equations (4.14) and (4.15) are satisfied automatically. However equation (4.16) becomes

$$(\pi^1_1-\pi^2_2)m^{12}-\pi^2_1m^{12}m^{11}+\pi^1_2m^{22}$$

$$+a_3(\pi^1_1+\pi^2_2-2\pi^3_3)=0. \tag{4.17}$$

The π^s_t can be obtained, as before, from (2.18). In the Ryan and Shepley [15] parametrization the trace free part of π^s_t, i.e. p_{ij}, is given by

$$6p_{ij}=e^{-\phi\kappa^3}[\alpha_1p_++\alpha_2p_-+\alpha_3\frac{3p_\phi}{\sinh(2\sqrt{3}\beta_-)}]e^{\phi\kappa^3}, \tag{4.18}$$

with α_1 and α_2 as before and

$$\alpha_3=\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}$$

It follows from (4.18) that (4.17) can be written as

$$3a_3p_+ - 3\alpha p_\phi - 3\lambda\coth(2\sqrt{3}\beta_-)p_\phi + \sqrt{3}\gamma p_- = 0,$$

where we have used the spin coefficients previously defined. Now we can check that the Hamiltonian (4.13) yields equations of motion which satisfy the space constraint equation (4.19) at all times. This we can do by taking the time derivative of (4.19), that is

$$3a_3\dot{p}_+ + 2\sqrt{3}\gamma\dot{p}_- - [3\alpha + 3\lambda\coth(2\sqrt{3}\beta_-)]\dot{p}_\phi$$
$$+6\sqrt{3}\lambda\mathrm{csch}^2(2\sqrt{3}\beta_-)p_\phi\dot{\beta}_- + [6\gamma\coth(2\sqrt{3}\beta_-)p_\phi - 4\sqrt{3}p_-]\dot{\phi}, \tag{4.20}$$

where one uses $\frac{d\lambda}{d\phi} = 2\gamma$ and $\frac{d\gamma}{d\phi} = -2\lambda$. Substituting Hamilton's equations as obtained from (4.13) for $\dot{p}_+, \dot{p}_-, \dot{p}_\phi, \dot{\beta}_-$ and $\dot{\phi}$ into (4.20), one verifies that it is indeed zero.

V. CONCLUSIONS

In this paper we have investigated the causes why the variational principles for Class B Bianchi models breaks down. We have identified as the basic cause of the problem that the divergence theorem in a non-coordinated bases does not have the usual form. The corrected form of the divergence theorem has been derived, and we have shown that applying the usual ADM reduction to the Einstein action for Bianchi models in combination with this corrected form of the divergence theorem gives the right Einstein field equations for Class B models.

The above formalism was used to construct a Hamiltonian formalism for diagonal (β diagonal) and symmetric (β with one off-diagonal term) vacuum Class B models.

In the diagonal case we found that the models of Types III, V and $\mathrm{VI}_{h\neq-1}$ are consistent in the sense that the space constraint equation is satisfied at all times. However, Types

IV and $\mathrm{VII}_{h\neq 0}$ are not. The problem in the latter two is that although a diagonal metric is inserted directly into the Einstein action, R_{12} is not identically zero and therefore the $R_{12} = 0$ equation is lost.

For the symmetric case we showed that R_{13} and R_{23} automatically vanish as a result of introducing a symmetric metric. Furthermore, all vacuum Class B models yield equations of motion which are consistent with the space constraint equation in the sense previously mentioned.

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